HEAT TRANSFER AT A WEDGE IN A HYPERSONIC LOW-DENSITY GAS STREAM

Yu. A. Koshmarov

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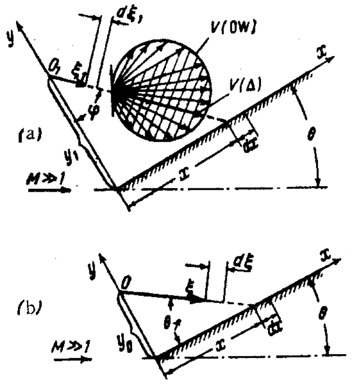
Results are presented of an approximate analysis of steady heat transfer on a sharp thin ($\sin\theta \approx \theta < 1$) strongly cooled ($t_w \ll 1$) wedge, washed by a hypersonic ($M \gg 1$) gas stream at zero angle of attack under almost free-molecule conditions. Dimensionless parameters on which the heat transfer depends have been established; approximate formulas for estimation purposes are given.

The present analysis is not a rigorous quantitative theory and its results should be regarded only as estimates; its conclusions may be useful for experimental planning and for generalization of test data. The method used here is analogous to that applied in [1] in analyzing flow over a flat plate.

1. **Initial statement and assumptions.** The discussion is based on a coordinate system tied to the wedge (see figure). Because the flow is symmetrical relative to the plane bisecting the wedge vertex angle, only the upper half is examined. For brevity the molecules of the undisturbed flow are designated O-molecules below.



Those leaving the wedge surface (reflected) without collision with molecules of other types, are called W-molecules. As a result of collisions, O- and W-molecules form OW-molecules. When OW-molecules collide with O- and W-molecules, secondary collision molecules are formed. The examination is limited to conditions where secondary collision molecules have a small probability of striking the wedge, i. e., heat transfer is studied in the flow regime when we need consider only the first collisions of incoming and reflected molecules.

It is assumed that the W-molecules form a monokinetic group, directed normal to the wedge surface, the velocity V(W) of the group being determined by the temperature of the wedge surface and corresponding to the condition of total accommodation of molecules striking the wedge. This assumption may be regarded as a crude schematization of diffuse reflection, according to which the majority of reflected molecules leave the surface almost normally (in conformity with the cosine law). The assumption of total accommodation will evidently be satisfied in many cases of practical interest. If necessary, however, partial accommodation may be allowed for approximately by special designation of the velocity of the W-molecules.

In the calculation of intermolecular collisions, the assumption is made that the gas molecules are perfectly smooth elastic spheres of identical mass. The molecular velocities after collision are determined by the laws of collision of elastic spheres [2]. Moreover, because of the inequality $t_w \ll 1$ and $M \gg 1$, it is assumed, in calculating the probable velocities of OW-molecules, that the W-molecules are at rest, while the O-molecules move with the velocity of the undisturbed stream, V. In that case the OW-molecules are dispersed from some arbitrary point in conformity with the cosine law for the velocity values. The ends of the OW-molecule probable velocity vectors V(OW), drawn from the point examined in space, form a spherical surface (see figure). One group of the OW-molecules formed close to the wedge surface is directed towards the wedge (figure; shaded part of vector diagram), while the other travels into space away from the wedge. In determining the energy carried to the wedge surface by OW-molecules, the average kinetic characteristics of molecules directed towards the wedge are used. Averaging reduces to replacing the flux of OW-molecules directed toward the wedge from the point in question by a monokinetic beam or ray, in which the velocity of the molecules is equal to the mean square of the probable velocities directed towards the wedge. The molecules of this beam are called Δ-molecules below. From the geometry, and the condition that any of the OW-molecule velocities shown on the vector diagram (figure a) is equally probable, we may calculate the fraction of the total number of OW-molecules formed at the point examined representing Δ-molecules (δ); the velocity ($V(\Delta)$) of the Δ-molecules; the angle between the normal drawn from the point examined to the wedge surface and the direction of velocity $V(\Delta)$ (angle φ)

$$\delta \approx \frac{1}{2}(1+\theta), \qquad V(\Delta) \approx \frac{1}{\sqrt{2}} V\left(1+\frac{\theta}{\pi}\right), \qquad \varphi \approx \frac{\pi}{4}. \quad (1.1)$$

Here θ is the half-angle at the wedge vertex. In deriving (1.1) we used the condition that θ is small (enabling us to neglect terms of order θ^2 in comparison with terms of order unity).

The number of collisions of O- and W-molecules has been determined by introducing a mean free path for W-molecules. The beam of W-molecules becomes attenuated with increased distance from the wedge surface due to collisions of W-molecules with S-molecules existing in the space above the wedge. Under the conditions examined there is a finite number of groups in the space above the wedge into which the S-molecules may be divided; they are mainly O-molecules and OW-molecules. The mean free path of W-molecules in an atmosphere of S-molecules is determined from the formula [3]

$$\lambda\,(\mathrm{W}) = V\,(\mathrm{W})\,/\,\omega\,(\mathrm{S})\,F\,(\mathrm{WS})\,V\,(\mathrm{WS})\,. \tag{1.2}$$

Here ω (S) is the number of S-molecules in unit volume; V (WS) is the mean relative velocity of W- and S-molecules before collision; F (WS) is the cross section for collisions of W- and S-molecules. In the first approximation the number density of S-molecules may be assumed constant and equal to that in the undisturbed stream, while the relative velocity V (WS) ≈ V. If it is assumed that the cross section does not depend on the relative velocity of the colliding molecules, then, taking account of the above remarks, it follows from (1.2) that

$$\lambda\,(\mathrm{W}) \approx \lambda t_w^{1/2}\,. \tag{1.3}$$

If it assumed that the collision cross section is inversely proportional to the relative velocity of the colliding molecules [4], it follows from (1.2) that

$$\lambda\,(\mathrm{W}) \approx \lambda t_w^{1/2} M \qquad (t_w = T_w/T_0)\,. \tag{1.4}$$

Here T_W is the temperature of the wedge surface (identical everywhere); T_0 is the adiabatic stagnation temperature of the stream; λ is the mean free path of molecules in the undisturbed stream; and M is the Mach number of the undisturbed stream. The number of W-molecules knocked out of the monokinetic beam through collisions, at an arbitrary point above the wedge surface, may be expressed in terms of λ(W) as follows [3]:

$$Q\,(\mathrm{W}) = \frac{n\,(\mathrm{W})}{\lambda\,(\mathrm{W})} \exp \frac{y}{\lambda\,(\mathrm{W})}\,. \tag{1.5}$$

Here Q(W) is the number of molecules knocked out from the W-group (i.e., the number of collisions in unit time and unit volume); n(W) is the number of W-molecules leaving in unit time from unit surface of the wedge at the point x.

In conformity with the initial conditions, it is assumed that the Δ-molecules do not collide with molecules of other groups.

In calculating the number of O-molecules reaching the wedge surface, use was made of the method of fictitious angle of attack [5]. The fictitious angle of attack θ_f is understood to be the angle at which the flux of O-molecules with velocity V directed from the undisturbed region to the area of the wedge under examination, without taking thermal velocities into account, will be equal to the flux to this area in free-molecular flow, allowing for the thermal velocities of the O-molecules

$$\sin\theta_f =$$

$$= \sin\theta \left\{ \frac{1}{2} \left(1 + \operatorname{erf} \sqrt{{}^1/{}_2\,\gamma}\, M \sin\theta\right) + \frac{\exp\left(-{}^1/{}_2\, M^2 \sin^2\theta\right)}{\sqrt{2\pi\gamma}\, M \sin\theta} \right\}. \tag{1.6}$$

2. Number of O- and OW-molecules incident on the wedge. Examination of the O-molecule balance in the volume element (1 × dξ) in the ray ξ, directed at angle θ_f to an arbitrary element of area (1 × dx) on the wedge surface (figure b) leads to the equation

$$V \frac{\partial \omega\,(\mathrm{O})}{\partial \xi} \approx -Q\,(\mathrm{W})\,. \tag{2.1}$$

Here ω(O) is the number of O-molecules in unit volume. In constituting the balance it is assumed, in accordance with the statement of the problem, that collisions above the wedge surface are mainly those of O- and W-molecules.

Calculation of the number of O-molecules incident on area (1 × dx) reduces to integration of (2.1). The quantity n(W) entering into Q(W) is an unknown function of ξ. As a good enough first approximation when calculating the number of O-molecules reaching area (1 × dx), we may assume for the conditions in question that n(W) is equal to the value occurring in free-molecular flow [1]. This assumption is also used in calculating the number of OW-molecules incident on the area (1 × dx) (see below). From this assumption and the condition that dξ = −dy/sin θ_f, it follows from (2.1) that

$$\frac{n\,(\mathrm{O})}{n_m} = 1 + \int_{\beta_0}^{0} \exp\,(-\beta)\,d\beta = \exp\,(-\alpha \operatorname{tg} \theta_f)\,,$$

$$\alpha = \frac{x}{\lambda\,(\mathrm{W})}, \qquad \beta = \frac{y}{\lambda\,(\mathrm{W})}, \qquad \beta_0 = \alpha \operatorname{tg} \theta_f\,. \tag{2.2}$$

Here n_m is the number of molecules incident on unit area of the wedge in unit time in free-molecular flow; n(O) is the number of O-molecules incident on unit area of the wedge at point x in unit time. Examination of the Δ-molecule balance in the volume element (1 × dξ_1) in the ray ξ_1 directed at angle φ to the area (1 × dx) on the wedge surface (figure a) leads to the equation

$$V\,(\Delta) \frac{\partial \omega\,(\Delta)}{\partial \xi_1} = 2Q\,(\mathrm{W})\,\delta\,. \tag{2.3}$$

Here $\omega(\Delta)$ is the number density of Δ-molecules; 2Q(W) is the total number of O-molecules formed in unit volume near the point in space being examined (the number of OW-molecules created is equal to the number of collisions of O- and W-molecules). Taking (1.1) into account, we have from (2.3)

$$\frac{n\,(\Delta)}{n_m} \approx (1 + \theta)\,[1 - \exp\,(-\alpha)]\,. \tag{2.4}$$

Here n (Δ) is the number of OW-molecules incident on unit area of the wedge. The number of W-molecules leaving unit area of the wedge at the point x may now be determined from the condition

$$n\,(\mathrm{W}) = n\,(\mathrm{O}) + n\,(\Delta)\,. \tag{2.5}$$

The value of n(W) found in this way may be used in seeking the second approximation for n(O) and n(Δ).

3. **Heat transfer to the wedge.** The heat flux is determined as the difference between the energy supplied by the O- and Δ-molecules and the energy removed by the W-molecules. The energy of monatomic molecules is the kinetic energy of their translational motion. If the molecules are diatomic, account must in general be taken of the internal energy of the molecules. For the conditions examined ($M \gg 1$; $t_W \ll 1$) the internal energy of the molecules, as well as the kinetic energy of translational motion of the W-molecules, is negligibly small in comparison with the kinetic energies of the O- and Δ-molecules. Taking this into account, and also (1.1), (2.2), (2.4) for small α and Z (the present theory is valid when α and Z are small), we obtain the following approximate formulas:

$$\frac{q_x}{q_m} \approx 1 + \frac{1}{2}\left(1 + \theta + \frac{2\theta}{\pi} - 2\,\mathrm{tg}\,\theta_f\right)\alpha,$$

$$\frac{q}{q_m} \approx 1 + \frac{1}{4}\left(1 + \theta + \frac{2\theta}{\pi} - 2\mathrm{tg}\,\theta_f\right)Z,$$

$$\left(q = \frac{1}{l}\int_0^l q_x\,dx,\ Z = \frac{l}{\lambda(\mathrm{W})}\right). \tag{3.1}$$

Here, and subsequently, q_m is the specific heat flux in free-molecular flow; q_x is the local specific heat flux; l is the length of the generator of the wedge. When $M\theta \gg 1$, it follows from (3.1), allowing for (1.6) that

$$\frac{q_x}{q_m} \approx 1 + \frac{1}{2}\left(1 + \frac{2\theta}{\pi} - \theta\right)\alpha, \qquad \frac{q}{q_m} \approx 1 + \frac{1}{4}\left(1 + \frac{2\theta}{\pi} - \theta\right)Z. \tag{3.2}$$

When $M\theta \ll 1$, from (3.1), allowing for (1.6) and the condition that θ and $2\theta/\pi$ are much less than the quantity $1/M$ (which follows from the inequality $M\theta \ll 1$), we derive the formulas

$$\frac{q_x}{q_m} \approx 1 + \frac{1}{2}\left[1 - \left(\frac{2}{\pi\gamma}\right)^{1/2}\frac{1}{M}\right]\alpha, \quad \frac{q}{q_m} \approx 1 + \frac{1}{4}\left[1 - \left(\frac{2}{\pi\gamma}\right)^{1/2}\frac{1}{M}\right]Z. \tag{3.3}$$

Determination of the mean free path of W-molecules according to (1.3) and (1.4), respectively, gives

$$\alpha = \frac{R_x}{M\,1.255\,\gamma^{1/2}t_w^{1/2}},\ Z = \frac{R}{M\,1.255\,\gamma^{1/2}t_w^{1/2}} = \left(\frac{\gamma-1}{2}\right)^{1/2}\left(\frac{T}{T_w}\right)^{1/2}\frac{M}{K},$$
$$\alpha = \frac{R_x}{M^2\,1.255\,\gamma^{1/2}t_w^{1/2}},\ Z = \frac{R}{M^2\,1.255\,\gamma^{1/2}t_w^{1/2}} = \left(\frac{\gamma-1}{2}\right)^{1/2}\left(\frac{T}{T_w}\right)^{1/2}\frac{1}{K}, \tag{3.4}$$

$$\left(\gamma = \frac{c_p}{c_v}, \quad R = \frac{\rho l V}{\mu}, \quad R_x = \frac{\rho x V}{\mu}, \quad K = \frac{\lambda}{l}\right). \tag{3.5}$$

Here ρ, T, μ are, respectively, density, thermodynamic temperature, and viscosity of the gas in the undisturbed stream; c_p and c_v are the isobaric and isochroic specific heats. In conclusion, it should be noted that: a) the heat fluxes in near free-molecular hypersonic flow over a wedge may noticeably exceed the values following from free-molecular flow theory (formulas (3.1)–(3.3)). When $M\theta \ll 1$, this effect diminishes with increase of θ (formula (3.2)), and when $M\theta \ll 1$ it diminishes with decreasing M (formula (3.3)); b) the values of the local and average Stanton numbers in near free-molecular flow depend, respectively, on α and Z (formulas (3.1)–(3.3)), and, in addition, when $M\theta \gg 1$, on θ (see (3.2)), and when $M\theta \ll 1$, on M (formula (3.3)).

The results of the present analysis agree with the predictions made in [4]. It is interesting to compare the second formula in (3.3), valid in the special case $M\theta = 0$, with the results of [5], in which a study was made in a more strict formulation of the heat transfer on a circular plate under conditions similar to those examined here, yielding the formula

$$\frac{q}{q_m} = 1 + 0.0725\,\frac{M}{K}\left(\frac{T}{T_w}\right)^{1/2}. \tag{3.6}$$

If the molecule collision cross section is considered constant and $\gamma = 1.4$, (3.3) takes the form

$$\frac{q}{q_m} \approx 1 + 0.111\,\frac{M}{K}\left(\frac{T}{T_w}\right)^{1/2}\left[1 - \left(\frac{2}{\pi\gamma}\right)^{1/2}\frac{1}{M}\right]. \tag{3.7}$$

For very large M the second term in the square brackets in (3.7) may be neglected, when an expression identical with (3.6) is obtained. A certain difference in numerical coefficients between (3.6) and (3.7) may be explained by inaccuracies due to the assumptions made, as well as to the fact that a circular plate was examined in [5], while here we have a plate of infinite extent. The conclusions of the present work are confirmed qualitatively by results of experimental investigations [6]. (The data of [6] were obtained at small values of the temperature factor $t_W = 0.5$–0.8, and may therefore be admitted here only to verify the qualitative conclusions).

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